On New Ideas of Nonlinearity in Quantum Mechanics

Vasyl Kovalchuk

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[1] V. Kovalchuk and J. J. Sławianowski:
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Symmetry, Integrability and Geometry: Methods and Applications
4, 046, 9 pages, 2008.

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[2] J. J. Sławianowski and V. Kovalchuk:
Schrödinger and related equations as Hamiltonian systems, manifolds of second-order tensors and new ideas of nonlinearity in quantum mechanics,
Reports on Mathematical Physics
65, 1, pp. 29–76, 2010.

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[3] J. J. Sławianowski and V. Kovalchuk:
Schrödinger equation as a hamiltonian system, essential nonlinearity, dynamical scalar product and some ideas of decoherence,
in: *Advances in Quantum Mechanics*,
Prof. Paul Bracken (Ed.),
ISBN: 978-953-51-1089-7,
InTech, Rijeka, pp. 81-103, 2013.

Quantum mechanics is plagued by paradoxes:

- decoherence,
- measurement process,
- reduction of the state vector.

Main concern is the linearity of the Schrödinger equation, which seems to be drastically incompatible with above-mentioned problems.

At the same time linearity works beautifully when:

- describing the unobserved unitary quantum evolution,
- finding the energy levels,
- in all statistical predictions.

Perhaps we deal here with a very sophisticated and delicate nonlinearity which becomes active and remarkable just in the process of interaction between quantum systems and "large" classical objects.

The main idea is:

- to analyze the Schrödinger equation and corresponding relativistic linear wave equations as usual self-adjoint equations of mathematical physics derivable from variational principles.

Lagrangian $\Rightarrow$ Hamiltonian:

- Legendre transformation for the Schrödinger and Dirac equations is uninvertible and leads to constraints in the phase space. Dirac formalism is the solution.

Incidentally, introducing the second-order time derivatives to dynamical equations, even as small corrections, regularizes Legendre transformation.

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In non-relativistic quantum mechanics there are certain hints suggesting just such a modification in the nano-scale physics.

[Kozlowski M., Marciak-Kozlowska J., *From quarks to bulk matter*, Hadronic Press, USA, 2001.]

[Marciak-Kozlowska J., Kozlowski M. *Schrödinger equation for nanoscience*, arXiv.org:cond-mat/0306699.]

Step 1:

The quantum Fourier equation which describes the heat (mass) diffusion on the atomic level has the following form:

$$\frac{\partial T}{\partial t} = \frac{\hbar}{m}\nabla^2 T.$$

If we take the substitution $t \to it/2$ and $T \to \psi$, then we end up with the free Schrödinger equation:

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi.$$

Step 2:

The complete Schrödinger equation with the potential term V after the reverse substitutions $t \to -2it$ and $\psi \to T$ gives us the parabolic quantum Fokker-Planck equation, which describes the quantum heat transport for $\Delta t > \tau$, where $\tau = \hbar/m\alpha^2 c^2 \sim 10^{-17}$ sec and $c\tau \sim 1$ nm, i.e.,

$$\frac{\partial T}{\partial t} = \frac{\hbar}{m}\nabla^2 T - \frac{2V}{\hbar}T.$$

Step 3:

For ultrashort time processes when $\Delta t < \tau$ one obtains the generalized quantum hyperbolic heat transport equation:

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{\hbar}{m} \nabla^2 T - \frac{2V}{\hbar} T.$$

Step 4:

This leads us to the second-order modified Schrödinger equation

$$2\tau\hbar \frac{\partial^2 \psi}{\partial t^2} + i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi.$$

in which the additional term describes the interaction of electrons with surrounding space-time filled with virtual positron-electron pairs.

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Analogy to superposition of Dirac and d'Alembert operators (KGD equation).

[Sławianowski J.J., Kovalchuk V. *Klein-Gordon-Dirac equation: physical justification and quantization attempts*, Rep. Math. Phys. **49** (2002), 249–257.]

The conceptual transition from special to general theory of relativity:

- the metric tensor looses its status of the absolute geometric object and becomes included into degrees of freedom (gravitational field).

In our treatment:

- the Hilbert-space scalar product becomes a dynamical quantity which satisfies together with the state vector the system of differential equations.

The main idea:

- there is no fixed scalar product metric!
- the dynamical term of Lagrangian describing the self-interaction of the metric is invariant under the total group $\mathrm{GL}(n, \mathbb{C})$.

The natural metric of this kind:

- introducing to the theory a very strong nonlinearity which induces also the effective nonlinearity of the wave equation even if there is no “direct nonlinearity” in it.

Strong nonlinearity prevents us from finding a rigorous solution.

But some partial results are possible:

- if we fix the behaviour of wave function to some simple form, then for the scalar product behaviour there are rigorous exponential solutions (including infinitely growing/exponentially decaying in the future — some decay/reduction phenomena).

Two kinds of degrees of freedom (dynamical variables):

- wave function,
- scalar product.

They are mutually interacting.

N-level quantum system:

We can define the "wave function" of the n-level quantum system as a following n-vector: Let us take a set of n elements and some function ψ defined on it, i.e.,

$$\psi = \begin{bmatrix} \psi^1 \\ \vdots \\ \psi^n \end{bmatrix}, \qquad \psi^a = \psi(a) \in \mathbb{C}.$$

Let H be a unitary space (n-dimensional "Hilbert space" $\mathbb{C}^n$) with the scalar product

$$G : H \times H \to \mathbb{C},$$

which is a sesquilinear hermitian form.

The general Lagrangian:

$$\begin{aligned} L &= \alpha_1 i G_{\bar{a}b}\left(\overline{\psi}^{\bar{a}}\dot{\psi}^b - \dot{\overline{\psi}}^{\bar{a}}\psi^b\right) + \alpha_2 G_{\bar{a}b}\dot{\overline{\psi}}^{\bar{a}}\dot{\psi}^b + \left[\alpha_4 G_{\bar{a}b} + \alpha_5 H_{\bar{a}b}\right]\overline{\psi}^{\bar{a}}\psi^b \\ &+ \alpha_3\left[G^{b\bar{a}} + \alpha_9\overline{\psi}^{\bar{a}}\psi^b\right]\dot{G}_{\bar{a}b} + \Omega[\psi, G]^{d\bar{c}b\bar{a}}\dot{G}_{\bar{a}b}\dot{G}_{\bar{c}d} - \mathcal{V}(\psi, G), \end{aligned}$$

where

$$\begin{aligned}\Omega[\psi, G]^{d\bar{c}b\bar{a}} &= \alpha_6 \left[G^{d\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^d\right] \left[G^{b\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^b\right] \\ &+ \alpha_7 \left[G^{b\bar{a}} + \alpha_9 \overline{\psi}^{\bar{a}} \psi^b\right] \left[G^{d\bar{c}} + \alpha_9 \overline{\psi}^{\bar{c}} \psi^d\right] + \alpha_8 \overline{\psi}^{\bar{a}} \psi^b \overline{\psi}^{\bar{c}} \psi^d, \\ \Omega[\psi, G]^{d\bar{c}b\bar{a}} &= \Omega[\psi, G]^{b\bar{a}d\bar{c}},\end{aligned}$$

and the potential $\mathcal{V}$ can be taken, for instance, in the following quartic form:

$$\mathcal{V}(\psi, G) = \varkappa \left(G_{\bar{a}b} \overline{\psi}^{\bar{a}} \psi^b\right)^2 .$$

The first and second terms (those with α_1 and α_2) describe the free evolution of wave function ψ while G is fixed. The Lagrangian for trivial part of the linear dynamics (those with α_4) can be also taken in the more general form $f\left(G_{\bar{a}b} \overline{\psi}^{\bar{a}} \psi^b\right)$, where $f : \mathbb{R} \to \mathbb{R}$. The term with α_5 corresponds to the Schrödinger dynamics while G is fixed and then

$$H^a{}_b = G^{a\bar{c}} H_{\bar{c}b}$$

is the usual Hamilton operator. If we properly choose the constants α_1 and α_5, then we obtain precisely the Schrödinger equation. The dynamics of the scalar product G is described by the terms linear and quadratic in the time derivative of G. In the above formulae $\overline{\psi}^{\bar{a}} = \overline{\psi^a}$ denotes the usual complex conjugation and α_i, $i = \overline{1, 9}$, and $\varkappa$ are some constants.

The equations of motion:

$$\begin{aligned}
\frac{\delta L}{\delta\overline{\psi}^{\bar a}} &= \alpha_2 G_{\bar a b}\ddot{\psi}^b + \left(\alpha_2\dot G_{\bar a b} - 2\alpha_1 i G_{\bar a b}\right)\dot\psi^b - 2\alpha_8 \dot G_{\bar a b}\psi^b \dot G_{\bar c d}\overline{\psi}^{\bar c}\psi^d \\
&- 2\alpha_9\left(\alpha_6 \dot G_{\bar a d}\dot G_{\bar c b} + \alpha_7 \dot G_{\bar a b}\dot G_{\bar c d}\right)\psi^b\left(G^{d\bar c} + \alpha_9\overline{\psi}^{\bar c}\psi^d\right) \\
&+ \left[\left(2\varkappa G_{\bar c d}\overline{\psi}^{\bar c}\psi^d - \alpha_4\right)G_{\bar a b} - \alpha_5 H_{\bar a b} - \left[\alpha_3\alpha_9 + \alpha_1 i\right]\dot G_{\bar a b}\right]\psi^b = 0, \\
\frac{\delta L}{\delta G_{\bar a b}} &= 2\Omega[\psi, G]^{b\bar a d\bar c}\ddot G_{\bar c d} + 2\dot\Omega[\psi, G]^{b\bar a d\bar c}\dot G_{\bar c d} + \left(2\varkappa G_{\bar c d}\overline{\psi}^{\bar c}\psi^d - \alpha_4\right)\overline{\psi}^{\bar a}\psi^b \\
&+ 2G^{d\bar a}\left[\alpha_6 G^{b\bar e}\left(G^{f\bar c} + \alpha_9\overline{\psi}^{\bar c}\psi^f\right) + \alpha_7 G^{b\bar c}\left(G^{f\bar e} + \alpha_9\overline{\psi}^{\bar e}\psi^f\right)\right]\dot G_{\bar c d}\dot G_{\bar e f} \\
&- \alpha_2\dot{\overline{\psi}}^{\bar a}\dot\psi^b + \left[\alpha_3\alpha_9 + \alpha_1 i\right]\dot{\overline{\psi}}^{\bar a}\psi^b + \left[\alpha_3\alpha_9 - \alpha_1 i\right]\overline{\psi}^{\bar a}\dot\psi^b = 0,
\end{aligned}$$

where

$$\begin{aligned}
\dot\Omega[\psi, G]^{b\bar a d\bar c} &= \alpha_8\left(\dot{\overline{\psi}}^{\bar a}\psi^b\overline{\psi}^{\bar c}\psi^d + \overline{\psi}^{\bar a}\dot\psi^b\overline{\psi}^{\bar c}\psi^d + \overline{\psi}^{\bar a}\psi^b\dot{\overline{\psi}}^{\bar c}\psi^d + \overline{\psi}^{\bar a}\psi^b\overline{\psi}^{\bar c}\dot\psi^d\right) \\
&+ \alpha_6\alpha_9\left(\left[\dot{\overline{\psi}}^{\bar a}\psi^d + \overline{\psi}^{\bar a}\dot\psi^d\right]\left[G^{b\bar c} + \alpha_9\overline{\psi}^{\bar c}\psi^b\right] + \left[\dot{\overline{\psi}}^{\bar c}\psi^b + \overline{\psi}^{\bar c}\dot\psi^b\right]\left[G^{d\bar a} + \alpha_9\overline{\psi}^{\bar a}\psi^d\right]\right) \\
&+ \alpha_7\alpha_9\left(\left[\dot{\overline{\psi}}^{\bar a}\psi^b + \overline{\psi}^{\bar a}\dot\psi^b\right]\left[G^{d\bar c} + \alpha_9\overline{\psi}^{\bar c}\psi^d\right] + \left[\dot{\overline{\psi}}^{\bar c}\psi^d + \overline{\psi}^{\bar c}\dot\psi^d\right]\left[G^{b\bar a} + \alpha_9\overline{\psi}^{\bar a}\psi^b\right]\right) \\
&- \alpha_6\left[G^{d\bar e}G^{f\bar a}\left(G^{b\bar c} + \alpha_9\overline{\psi}^{\bar c}\psi^b\right) + G^{b\bar e}G^{f\bar c}\left(G^{d\bar a} + \alpha_9\overline{\psi}^{\bar a}\psi^d\right)\right]\dot G_{\bar e f} \\
&- \alpha_7\left[G^{b\bar e}G^{f\bar a}\left(G^{d\bar c} + \alpha_9\overline{\psi}^{\bar c}\psi^d\right) + G^{d\bar e}G^{f\bar c}\left(G^{b\bar a} + \alpha_9\overline{\psi}^{\bar a}\psi^b\right)\right]\dot G_{\bar e f}.
\end{aligned}$$

Pure dynamics for G:

The equations of motion for the pure dynamics of scalar product G while the wave function ψ is fixed are as follows:

$$
\begin{aligned}
\Omega[\psi, G]^{b\bar{a}d\bar{c}}\ddot{G}_{\bar{c}d} &= \left(\frac{\alpha_4}{2} - \varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^{d}\right)\overline{\psi}^{\bar{a}}\psi^{b} + \alpha_7 \dot{G}_{\bar{c}d}\dot{G}_{\bar{e}f}\gamma[\psi, G]^{d\bar{e}f\bar{c}b\bar{a}} \\
&+ \alpha_6 \dot{G}_{\bar{c}d}\dot{G}_{\bar{e}f}\left(\gamma[\psi, G]^{b\bar{e}f\bar{c}d\bar{a}} + \gamma[\psi, G]^{f\bar{a}d\bar{e}b\bar{c}} - \gamma[\psi, G]^{b\bar{e}d\bar{a}f\bar{c}}\right),
\end{aligned}
$$

where

$$
\gamma[\psi, G]^{f\bar{e}d\bar{c}b\bar{a}} = G^{f\bar{e}}G^{d\bar{c}}\left(G^{b\bar{a}} + \alpha_9\overline{\psi}^{\bar{a}}\psi^{b}\right).
$$

If we additionally suppose that $\alpha_4 = \alpha_8 = \alpha_9 = \varkappa = 0$, then the above expression simplifies significantly:

$$
\left(\alpha_6 G^{b\bar{c}}G^{d\bar{a}} + \alpha_7 G^{b\bar{a}}G^{d\bar{c}}\right)\left(\ddot{G}_{\bar{c}d} - \dot{G}_{\bar{c}f}G^{f\bar{e}}\dot{G}_{\bar{e}d}\right) = 0.
$$

Hence, the pure dynamics of the scalar product is described by the following equations:

$$
\ddot{G}_{\bar{a}b} - \dot{G}_{\bar{a}d}G^{d\bar{c}}\dot{G}_{\bar{c}b} = 0.
$$

Let us now demand that $\dot{G}G^{-1}$ is equal to some constant value E, i.e., $\dot{G} = EG$, then

$$\ddot{G} = E\dot{G} = E^2 G$$

and

$$\dot{G}G^{-1}\dot{G} = EGG^{-1}EG = E^2 G,$$

therefore our equations of motion are fulfilled automatically and the solution is as follows:

$$G(t)_{\bar{a}b} = \left(\exp(Et)\right)^{\bar{c}}{}_{\bar{a}} G_{0\bar{c}b}.$$

$$* \; * \; *$$

Similarly if we demand that $G^{-1}\dot{G}$ is equal to some other constant E', i.e., $\dot{G} = GE'$,

$$\ddot{G} = \dot{G}E'^2 = GE'^2, \qquad \dot{G}G^{-1}\dot{G} = GE'G^{-1}GE' = GE'^2,$$

then the equations of motion are also fulfilled and the solution is as follows:

$$G(t)_{\bar{a}b} = G_{0\bar{a}d} \left(\exp(E't)\right)^{d}{}_{b}.$$

The connection between these two different constants E and E' is written below:

$$\dot{G}(0) = \dot{G}_0 = G_0 E' = E G_0.$$

Usual and first-order modified Schrödinger equations:

The second interesting special case is obtained when we suppose that the scalar product G is fixed, i.e., the equations of motion are as follows:

$$\alpha_2\ddot{\psi}^a - 2\alpha_1 i\dot{\psi}^a + \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d\left[\psi, G\right] - \alpha_4\right)\psi^a - \alpha_5 H^a{}_b\psi^b = 0.$$

Then if we also take all constants of model to be equal to 0 except of the following ones:

$$\alpha_1 = \frac{\hbar}{2}, \qquad \alpha_5 = -1,$$

we end up with the well-known usual Schrödinger equation:

$$i\hbar\dot{\psi}^a = H^a{}_b\psi^b.$$

Its first-order modified version is obtained when we suppose that G is a dynamical variable and α_2 is equal to 0, i.e.,

$$\begin{aligned} i\hbar\dot{\psi}^a &= H^a{}_b\psi^b - \left[\frac{i\hbar}{2} + \alpha_3\alpha_9\right]G^{a\bar{c}}\dot{G}_{\bar{c}b}\psi^b + \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^d - \alpha_4\right)\psi^a \\ &- 2\alpha_8 G^{a\bar{c}}\dot{G}_{\bar{c}b}\psi^b\dot{G}_{\bar{e}d}\overline{\psi}^{\bar{e}}\psi^d - 2\alpha_9 G^{a\bar{c}}\left(\alpha_6\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}b} + \alpha_7\dot{G}_{\bar{c}b}\dot{G}_{\bar{e}d}\right)\psi^b\left(G^{d\bar{e}} + \alpha_9\overline{\psi}^{\bar{e}}\psi^d\right), \end{aligned}$$

$$
\begin{aligned}
2\Omega[\psi, G]^{b\bar{a}d\bar{c}}\ddot{G}_{\bar{c}d} &= \left[\frac{i\hbar}{2} - \alpha_3\alpha_9\right]\overline{\psi}^{\bar{a}}\dot{\psi}^{b} - \left[\frac{i\hbar}{2} + \alpha_3\alpha_9\right]\dot{\overline{\psi}}^{\bar{a}}\psi^{b} \\
&- 2G^{d\bar{a}}\left[\alpha_6 G^{b\bar{e}}\left(G^{f\bar{c}} + \alpha_9\overline{\psi}^{\bar{c}}\psi^{f}\right) + \alpha_7 G^{b\bar{c}}\left(G^{f\bar{e}} + \alpha_9\overline{\psi}^{\bar{e}}\psi^{f}\right)\right]\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}f} \\
&- \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^{d} - \alpha_4\right)\overline{\psi}^{\bar{a}}\psi^{b} - 2\dot{\Omega}[\psi, G]^{b\bar{a}d\bar{c}}\dot{G}_{\bar{c}d}.
\end{aligned}
$$

We can rewrite the above equation of motion for ψ in the following form:

$$
i\hbar\dot{\psi}^{a} = H_{\mathrm{eff}}{}^{a}{}_{b}\psi^{b},
$$

where the effective Hamilton operator is given as follows:

$$
\begin{aligned}
H_{\mathrm{eff}}{}^{a}{}_{b} &= H^{a}{}_{b} - \left[\frac{i\hbar}{2} + \alpha_3\alpha_9\right]G^{a\bar{c}}\dot{G}_{\bar{c}b} + \left(2\varkappa G_{\bar{c}d}\overline{\psi}^{\bar{c}}\psi^{d} - \alpha_4\right)\delta^{a}{}_{b} - 2\alpha_8 G^{a\bar{c}}\dot{G}_{\bar{c}b}\dot{G}_{\bar{e}d}\overline{\psi}^{\bar{e}}\psi^{d} \\
&- 2\alpha_9 G^{a\bar{c}}\left(\alpha_6\dot{G}_{\bar{c}d}\dot{G}_{\bar{e}b} + \alpha_7\dot{G}_{\bar{c}b}\dot{G}_{\bar{e}d}\right)\left(G^{d\bar{e}} + \alpha_9\overline{\psi}^{\bar{e}}\psi^{d}\right).
\end{aligned}
$$

Future research:

- What if we admit “dissipative” models, where the Schrödinger equation does possess some “friction-like” term? $\Rightarrow$ Some quantum models of dissipation.

Further investigation is required.

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Thank you for your attention!

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